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THE SYNCHRONIZATION OF OSCILLATORS WHICH INTERACT VIA A MEDIUM*

E.E. SHNOL

A system of N non-linear oscilators which influence one another only as a result of their action on a common medium, is considered. The stability of the synchronized or partially synchronized periodic oscillations in the system is discussed. Special attention is paid to the case when $N \gg 1$. The problem of the synchronization of such oscillators, which do not interact directly but only indirectly via a common medium, is not new /1, 2/. It is usually assumed that the interaction is weak, so that the oscillators only slightly change their frequency and shape. The term "synchronization" usually means one of two effects: 1) the establishment of identical oscillations (in shape and phase) in a system of identical oscillators, 2) the establishment of a common period of oscillation in a system of identical or structurally similar oscillators. Biological problems which lead to a synchronization problem are considered in /3, 4/.

1. The system and special features of the problem. We consider N objects of the same nature which have a stable selfexcited oscillatory mode in a band of fixed external conditions. The objects (oscillators) are located in a medium which they influence and thereby influence one another. We assume that the action of the oscillators on the medium is additive.

In biological applications, the number N of objects is usually very large, and the influence of one oscillator on the medium is very small. For instance, if we are speaking of the biological oscillations inherent in a living cell, the action of an individual cell on the medium is proportional to the ratio of the cell volume v to the volume V of the medium in which there are no cells.

In the elementary case of identical oscillators, the equations describing the system can be written as (the dot denotes differentiation with respect to time t)

$$s^{\cdot} = \frac{1}{N}\sum_{k=1}^{N} g(s, x_k); \quad x_k^{\cdot} = h(s, x_k); \quad k = 1, \ldots, N \tag{1.1}$$

$$g(s, x) = g^{(0)}(s) + \gamma g^{(1)}(s, x), \ \ \gamma = Nv/V \tag{1.2}$$

Here, the vector s refers to the medium and x to the oscillators (in general, $\dim s \neq \dim x$), $g^{(0)}(s)$ describes the change in the medium regardless of its "filling", and $g^{(1)}(s, x)$ is the influence of the oscillators on the medium. It is assumed below that (given a fixed "density" γ) $g(s, x)$ is independent of N.

In biological (as distinct from technical) applications, synchronization is usually of interest when it sets in rapidly (in a fairly small number of periods) and is preserved when

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the oscillator parameters differ substantially. This is possible provided that the interaction is not excessively weak. In particular, the oscillations of the global variable s must have significant amplitude (this condition is only necessary; it is also necessary that the oscillators be "sensitive" to oscillations of s).

To sum up, our problem has the following features: 1) the number N of oscillators is very large and we are only interested in mathematical assertions that remain (or become) meaningful as $N \to \infty$; 2) the interaction of the oscillations is not assumed to be weak.

In the case of strong interaction, individuality is largely lost. In particular, for sufficiently large γ, system (1.1) may not have periodic solutions. Clearly, not many general assertions can be made about system (1.1).

2. Elementary periodic modes in a system of identical oscillators. When all the $x_k(t)$ are the same, the solutions of system (1.1) can be found from the equations

$$s^{\cdot} = g(s, X), \quad X^{\cdot} = h(s, X) \tag{2.1}$$

The periodic solutions $z^\circ(t) = (s^\circ(t);\ X^\circ(t), \ldots, X^\circ(t))$ of system (1.1) will be called synchronous, and the set of solutions which differ only by a phase shift (correspond to the same trajectory), a synchronous mode.

The system of oscillators can be divided into a small number m of interior synchronous subsystems. For the most interesting case $m = 2$

$$x_i(t) = X_1(t),\ 1 \leqslant i \leqslant N_1;\ x_j(t) = X_2(t),\ N_1 < j \leqslant N \tag{2.2}$$

Then, $s(t)$, $X_1(t)$, and $X_2(t)$ satisfy the system

$$\begin{gathered} s^{\cdot} = \beta g(s, X_1) + (1 - \beta)\, g(s, X_2), \quad \beta = N_1/N \\ X_1^{\cdot} = h(s, X_1), \quad X_2^{\cdot} = h(s, X_2) \end{gathered} \tag{2.3}$$

The periodic solutions of a system (1.1) of type (2.2) will be called two-component.

Note 1. There are usually only a few synchronous modes (most commonly, only one). The two-component modes as a rule form a family which depends on the parameter β: system (2.3) usually has periodic solution $(s^\circ(t), X_1^\circ(t), X_2^\circ(t))$ for the range of β values: $\beta_1 < \beta < \beta_2$. As $N \to \infty$, the number of admissible β $(= k/N)$ increases without limit.

Below, in Sects.3-5, the term "stability" signifies "asymptotic orbital stability with respect to the linear approximation."

3. Summary of results. *Assertion 1.* The necessary and sufficient condition for the synchronous mode in system (1.1) to be stable for any $N > 1$ is that it be stable for $N = 2$.

Assertion 2. If the synchronous mode is stable, the size of its domain of attraction Ω is independent of N. (More precisely, the internal diameter of Ω does not tend to zero as $N \to \infty$.)

For large N, the usual definitions of stability "with respect to the initial data" become insufficient. Clearly, any initial (at $t = 0$) deviation of a small number of components (oscillators) is a weak effect for the system as a whole. On allowing such deviations, we obtain:

Assertion 3. If the synchronous periodic solution in system (1.1) is stable, it is stable in the wider sense (see the definition in Sect.5).

We shall now consider a more realistic problem. Let the oscillators differ somewhat, while having the same nature.

Assertion 4. Let system (1.1) have a stable synchronous mode l°. Then, there exists $\delta > 0$, independent of N, such that any similar system

$$s^{\cdot} = \frac{1}{N}\sum_{k=1}^{N} g_k(s, x_k), \quad x_k^{\cdot} = h_k(s, x_k)$$

$$\| g_k - g \|_1 \leqslant \delta, \quad \| h_k - h \|_1 \leqslant \delta, \quad k = 1, \ldots, N$$

has a stable periodic mode l, close to l°. Here, $\| F \|_1$ is the norm in C^1, i.e., (in the natural notation)

$$\| F(s, x) \|_1 = \max\left(| F | + | \partial F / \partial s | + | \partial F / \partial x | \right)$$

Assertions 1 and 3 are proved below; see Sect.6 concerning Assertions 2 and 4.

4. The stability of the synchronous mode. Our interest is in reasonably strong (exponential) stability, provided by the linear approximation. Here, as is always the case for periodic solutions of autonomous systems, we are talking of a tendency to an undisturbed trajectory, i.e., of orbital stability.

Theorem 1. The necessary and sufficient conditions for a periodic solution $z^\circ(t) = (s^\circ(t);$

$X^\circ(t), \ldots, X^\circ(t))$ of system (1.1) to be asymptotically orbitally stable with respect to the linear approximation for any $N > 1$ are:

a) stability (in the same sense) of the solution $(s^\circ(t), X^\circ(t))$ of system (2.1) (i.e., of system (1.1) with $N = 1$);

b) asymptotic stability with respect to the linear approximation of the solution $X = X^\circ(t)$ of the non-autonomous system

$$X^\cdot = h(s^\circ(t), X) \tag{4.1}$$

Proof. On linearizing system (1.1) in the solution $z^\circ(t)$ and putting $y = (\sum x_k)/N$, $\xi_k = x_k - y$, we obtain

$$s^\cdot = a(t)s + b(t)y, \quad y^\cdot = p(t)s + q(t)y \tag{4.2}$$

$$\xi_k^\cdot = q(t)\xi_k \tag{4.3}$$

System (4.2) is formally the same as the linearization of the "one-particle" system (2.1), while system (4.3) is the result of linearizing system (4.1). By condition a), any solution of system (4.2) has the form

$$s(t) = Cs^{\cdot\circ}(t) + r_1(t), \; y(t) = CX^{\cdot\circ}(t) + r_2(t) \tag{4.4}$$
$$|r_1(t)| \leqslant C_1e^{-\alpha t}, \quad |r_2(t)| \leqslant C_2e^{-\alpha t}, \quad \alpha > 0$$

($|s|$ and $|x|$ are fixed norms in spaces s and x respectively, and the constants C, C_1, C_2 depend on the chosen solution).

By condition b), for any solution of system (4.3) we have

$$|\xi(t)| \leqslant C_3e^{-\alpha t} \tag{4.5}$$

Hence, for any solution of system (1.1),

$$s(t) = Cs^{\cdot\circ}(t) + R_0, \quad x_k(t) = CX^{\cdot\circ}(t) + R_k, \; |R_j(t)| \leqslant Be^{-\alpha t} \tag{4.6}$$

Eqs.(4.6) mean that the solution $z^\circ(t)$ of system (1.1) is asymptotically orbitally stable with respect to the linear approximation. The sufficiency is proved.

The proof of necessity is just as simple: from (4.6) there follow (4.5) and (4.4) for s, y, ξ, which formally represent the conditions of the theorem.

Note 2. Assertion 1 of Sect.3 follows obviously from Theorem 1.

Note 3. (see Assertion 2 of Sect.3). In the conditions of Theorem 1, there exists δ_*, independent of N, such that, with

$$\delta < \delta_*, \quad |s(0) - s^\circ(0)| < \delta, \quad |x_k(0) - X^\circ(0)| < \delta$$

we have the inequalities

$$|s(t) - s^\circ(t+\tau)| \leqslant B\delta e^{-\alpha t}, \; |x_k(t) - X^\circ(t+\tau)| \leqslant B\delta e^{-\alpha t}, \; k = 1, \ldots, N$$

The analogue of Theorem 1 for the two-component mode is:

Theorem 1a. The necessary and sufficient conditions for stability of a periodic solution of type (2.2) of system (1.1) with any $N_1 > 1$ and $N_2 = N - N_1 > 1$ are: a) stability of the solution $(s^\circ(t), X_1^\circ(t), X_2^\circ(t))$ of system (2.3); b) asymptotic stability with respect to the linear approximation of peridic solutions $X_1^\circ(t)$ and $X_2^\circ(t)$ of system (4.1).

5. Stability of the synchronous mode in the wider sense.

Definition 1. A synchronous (of period T) solution $z^\circ(t)$ of system (1.1) ($s = s^\circ(t)$, $x_k = X^\circ(t)$; $k = 1, \ldots, N$) is N-stable if, given any $\varepsilon > 0$, there exists $\delta > 0$ (independent of N) with the following properties. Let $|s(0) - s^\circ(0)| < \delta$, $|x_k(0) - X^\circ(0)| < \delta$ for $k \leqslant \nu$, $\nu \geqslant N(1-\delta)$. Then, in any interval $(t_*, t_* + T)$, for the solution $z(t)$ $|s(t) - s^\circ(t+\tau)| < \varepsilon$, $|x_k(t) - X^\circ(t+\tau)| < \varepsilon$, $k = 1, \ldots, \nu$.

Commentary. 1°. Nothing is said in the definition about variables x_k with $k > \nu$: they can be strongly varied at $t = 0$ and varied to some extent for $t > 0$.

2°. It is not essential to consider time intervals equal to the period of oscillation; we can choose any (fixed) T.

3°. Note that τ depends on t_* as well as on the solution $z(t)$.

Theorem 2. Let $|g(s, x)| \leqslant B$ (for all s and x) and let the synchronous periodic solution $z^\circ(t)$ of system (1.1) be asymptotically orbitally stable with respect to the linear approximation. Then, this solution is also stable in the sense of definition 1.

The proof is based on the following lemma, which we quote without proof.

Lemma. Under the hypotheses of the theorem, there exists a Lyapunov function $L(z)$ which satisfies, for $\rho(z) \leqslant M$, the following estimates with constants, independent of N:

$$c_0\rho(z) \leqslant L(z) \leqslant C_0\rho(z), \quad \left|\frac{\partial L}{\partial z_j}\right| \leqslant C_1, \quad L^\cdot|_{(1.1)} \leqslant -\rho(z) \tag{5.1}$$

Here, $\rho(z)$ is the distance from point z to the closed trajectory $l = \{z^\circ(t)\}$ in the sense of the norm

$$\|z\| = \max(|s|;\ |x_1|, \ldots, |x_N|) \tag{5.2}$$

Notice that any stable (in the same sense) periodic solution $z(t)$ of a system admits of a Lyapunov function with estimates (5.1). The significance of the lemma is that, for system (1.1), the constants c_0, C_0, C_1, M are independent of N.

Proof of Theorem 2. A system (1.1) for n objects will be called a system S_n. We specify $\delta > 0$ and $\nu \geqslant N(1-\delta)$. We put $z = (s; x_1, \ldots, x_N)$, $\zeta = (s; x_1, \ldots, x_\nu)$. On a solution of system S_N the variables s and x_k with $k \leqslant \nu$ satisfy the equations

$$s^\cdot = \frac{1}{\nu}\sum_{\kappa=1}^{\nu} g(s, x_k) + r(t), \quad x_k^\cdot = h(s, x_k), \quad k = 1, \ldots, \nu \tag{5.3}$$

While the remainder term $r(t)$ depends on the chosen solution, we always have $|r(t)| \leqslant 2B\delta$.

Let $L(\zeta)$ be the Lyapunov function of the lemma for system S_ν. On evaluating the derivative of L in the light of system S_N (or what amounts to the same thing, in the light of system (5.3)), we obtain

$$L^\cdot|_{S_N} = L^\cdot|_{S_\nu} + \frac{\partial L}{\partial s} r(t) \leqslant -\rho(\zeta) + C_2\delta \quad (C_2 = 2C_1B)$$

Here, $\rho(\zeta)$ is the distance of point ζ to the periodic solution $\zeta^\circ(t)$ of system S_ν ($s = s^\circ(t)$, $x_k = X^\circ(t)$, $k = 1, \ldots, \nu$).

Hence, $L^\cdot \leqslant 0$ for $C_2\delta \leqslant \rho(\zeta) \leqslant M$, or

$$L^\cdot|_{(5.3)} \leqslant 0 \text{ for } C_3\delta \leqslant L \leqslant M_1 \ (C_3 = C_0C_2,\ M_1 = c_0M) \tag{5.4}$$

We choose the initial point $z(0)$ for the solution of system S_N in such a way that $\|\zeta(0) - \zeta^\circ(0)\| \leqslant \delta$; then, $\rho(\zeta(0)) \leqslant \delta$, $L(\zeta(0))| \leqslant C_0\delta$. By (5.4), $L(\zeta(t)) \leqslant C_4\delta$ for all $t \geqslant 0$ ($C_4 = \max(C_0, C_3)$) and $\rho(\zeta(t)) \leqslant C_5\delta$, $C_5 = C_4/c_0$. We fix t_*; for some τ, we have $\|\zeta(t_*) - \zeta^\circ(t_* + \tau)\| \leqslant C_5\delta$. To estimate $w(t) = \zeta(t) - \zeta^\circ(t+\tau)$ for $t_* \leqslant t \leqslant t_* + T$, we note that $\zeta(t)$ and $\zeta^\circ(t)$ satisfy system (5.3) ($\zeta(t)$ with some $r(t)$, and $\zeta^\circ(t)$ with $r(t) \equiv 0$). As usual,

$$w^\cdot = A(t)w + R(t), \ \|A(t)\| \leqslant \max\|df/d\zeta\| = D,$$
$$\|R\| \leqslant 2B\delta$$

Here, $f(\zeta)$ is the right-hand side of system S_ν, the matrix norm corresponds to the vector norm (5.2), and the maximum is taken over a bounded domain which contains the domain $\rho(\zeta) \leqslant M$.

For system S_N D is independent of n. Hence it follows that $\|w(t)\| \leqslant C_6\delta$ for $t_* \leqslant t \leqslant t_* + T$, or $|s(t) - s^\circ(t+\tau)| \leqslant C_6\delta$, $|x_k(t) - X^\circ(t+\tau)| \leqslant C_6\delta$, $k = 1, \ldots, \nu$ (C_6 depends on T, but is independent of N).

In short, we can choose $\delta = \varepsilon/C_6$, the theorem is proved.

6. Addenda and notes. 1°. For small γ (see (1.1)) the oscillators interact weakly, and the theorems originating from Poincaré come into effect (see the references in /1, 2/). We assume that a) the system $s^\cdot = g^{(0)}(s)$ has an asymptotically stable equilibrium position s_*; b) the system $x^\cdot = h(s_*, x)$ has an asymptotically stable limiting cycle l. Then, with $\gamma \ll 1$, there exists in system (1.1) a synchronous mode (in which $s(t)$ is close to s_*, while $\{x(t)\}$ is close to l; and there exists a two-component mode (2.2) or period T, for which $N_1 = N/2$, $X_2(t) = X_1(t + T/2)$. Both these modes may be stable or unstable. Notice that, with $\beta = 1/2$, $X_1(t)$ and $X_2(t)$ come to be significantly different (while if $\beta \neq 1/2$, their diference becomes essential).

2°. Assertions 2 and 4 of Sect.3 (and the lemma of Sect.5) differ from the standard assertions only in the fact that they are independent of N and the fact that no new ideas are needed for their proof. The usual proofs for system $z^\cdot = f(z)$ (see e.g., /5/) serve the purpose if we use the fact that, for system (1.1):

a) $\|df(\xi)\| = \|\Sigma(\partial f/\partial z_j)\xi_j\| \leqslant C_1\|\xi\|$ (in the same way as for d^2f)

b) for the general solution $z = \Phi(t, \zeta)$ $(\Phi(0, \zeta) = \zeta)$

$$\|d_\zeta\Phi\| = \|\Sigma(\partial\Phi/\partial\zeta_j)\xi_j\| \leqslant B_1\|\xi\| \quad \text{(similarly for } d^2\Phi)$$

Here, C_k and B_k are independent of N if the norm (5.2) is used.

3°. We shall explain what has been said by taking the example of an analogue of Assertion 2 for a (simpler) case of equilibrium position.

Theorem. Let $z_* = (s_*; X_*, \ldots, X_*)$ be a symmetric equilibrium position of system (1.1), which is asymptotically stable with respect to the linear approximation. Then, with $\|z(0) - z_*\| < \delta$

$$\|z(t) - z_*\| \leqslant B\|z(0) - z_*\| e^{-\alpha t}, \quad \alpha > 0$$

(δ, α and B are independent of N).

Sketch of the proof. Let $z_* = 0$. We write system (1.1) as (see Sect.4)

$$s^{\cdot} = as + by + r_0(z), \quad x_k^{\cdot} = ps + qx_k + r_k(z)$$

or $z^{\cdot} = Az + R(z)$, $\| R \| \leqslant C_2 \| z \|^2$. By hypothesis, the eigenvalues of the matrices q and $\left\| \begin{matrix} a & b \\ p & q \end{matrix} \right\|$ satisfy the inequality $\operatorname{Re} \lambda \leqslant -\alpha < 0$. Hence $\exp(tA) \| \leqslant C \exp(-\alpha t)$ (C is independent of N). After this, from the identity

$$z(t) = e^{tA} z(0) + \int_0^t e^{(t-\tau)A} R(z(\tau))\, d\tau$$

we obtain $\| z(t) \| \leqslant 2C \| z(0) \| \exp(-\alpha t)$ for $\| z(0) \| < \delta_*$ (δ_* depends on C_2 but not on N).

4°. With $N \gg 1$, the removal (or addition) of a few oscillators represents a small disturbance of the system. The stability of such an operation, which changes the dimensionality of the system phase space, is not usually considered in the theory of differential equations. We maintain this custom in Sect.5, though in essence Theorem 2 remains valid in such cases.

5°. An analogoue of Theorem 2 holds for two-component modes. These make an explicit appearance as an entire family. A strong initial deviation of a few oscillators can transfer them from one subsystem to another; a small disturbance β is a special case of what is allowed by Theorem 2.

6°. Instead of Definition 1 of Sect.5 we can use the following more traditional-sounding definition:

Definition 2. The synchronous solution $z^{\circ}(t)$ of system (1.1) is N-stable if it is orbitally stable uniformlv with respect to N when we use the norm

$$\| z \| = | s | + \frac{1}{N} \sum_{k=1}^{N} | x_k |$$

While Definitions 1 and 2 are not formally equivalent, they are equivalent for the problem considered here, if an unbounded increase in x_k is not allowed (e.g., we put $h(s, x) \equiv 0$ outside the bounded domain Ω). E.E. Vol'kov drew the author's attention to the problems whose mathematical evolution is dicussed in the present paper, and the author thanks him for numerous discussions.

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